



The Open
University

MST124

Essential mathematics 1

Exercise Booklet 7

1 Derivatives of further standard functions

Exercise 1

Find the derivatives of the following functions.

(a) $f(x) = \cos x - 2 \sin x$

(b) $g(t) = t^2 + \tan t$

(c) $h(a) = 3(1 + \cos a)$

(d) $y = \frac{4}{x} + \sin x$

Exercise 2

Find the derivatives of the following functions.

(a) $f(x) = \sin x + 2 \ln x$

(b) $g(t) = 2e^t + \cos t$

(c) $y = 5(2 \sin x - 3)$

(d) $w = \frac{x^2}{2} + 2 \tan x$

(e) $p(r) = 3 \left(\tan r + \frac{1}{r^2} \right)$

2 Finding derivatives of more complicated functions

Exercise 3

Use the product rule to differentiate each of the following functions.

(a) $y = x^3 \ln x$ (b) $m(x) = e^x \sin x$

(c) $v = t^2 \sin t$

Exercise 4

Use the product rule to differentiate each of the following functions.

(a) $f(x) = (x^3 - 8x^{4/3} + 2) \cos x$

(b) $g(t) = (2t^4 + \sin t) \tan t$

(c) $h(x) = \left(5x^7 - x^2 + 5 - \frac{2}{x} \right) e^x$

(d) $f(\theta) = \theta^6 \ln \theta$

(e) $y = (1 - x^3) \ln x$

Exercise 5

Differentiate the function $p(x) = \frac{e^x}{x^2}$ by writing $p(x)$ as a product.

Exercise 6

Find the gradient of the graph of the equation $y = x^3 \ln x$ at the point with x -coordinate 1.

Exercise 7

Use the quotient rule to differentiate each of the following functions.

(a) $f(x) = \frac{x^7 + 1}{x^4 + 1}$ (b) $g(x) = \frac{4x + e^x}{x^8 - 2}$

(c) $y = \frac{\sin x}{e^x}$ (d) $m = \frac{\ln x}{x}$

Exercise 8

(a) Differentiate the function

$$y = 3x - \frac{4}{x} + \frac{1}{x^2}.$$

(b) Show that the rule of this function can also be written as

$$y = \frac{3x^3 - 4x + 1}{x^2},$$

and confirm your answer to part (a) by using the quotient rule.

Exercise 9

Use the quotient rule to differentiate each of the following functions.

$$(a) \ p(x) = \frac{x^2 - 1}{x^2 + 1} \quad (b) \ z(y) = \frac{2 \sin y}{y^2 - 4y + 3}$$

Exercise 10

For each of the following equations, write y as a function of u , where u is a function of x , and hence use the chain rule to find dy/dx .

$$(a) \ y = e^{x^2/2} \quad (b) \ y = \sin(x^3) \\ (c) \ y = \cos(\sqrt{x}) \quad (d) \ y = \sqrt{\cos x} \\ (e) \ y = \tan(\ln x)$$

Exercise 11

Use the chain rule to find the derivative of each of the following functions.

$$(a) \ y = \sqrt{3x^2 + 2x + 1} \quad (b) \ w = e^{-r^2}$$

Exercise 12

Find the derivative of each of the following functions.

$$(a) \ f(x) = e^{3x} \sin(2x) \quad (b) \ g(t) = \frac{3t^2}{\ln(4t)} \\ (c) \ y = (x^3 + 2x - 1) \cos(5x) \\ (d) \ z = \sin(e^{3t})$$

Exercise 13

Differentiate the following functions.

$$(a) \ f(t) = \cos^2(3t) \ln(5t) \\ (b) \ f(x) = \frac{e^{2x}}{(x^2 + 1)^2} \\ (c) \ f(x) = \sin((x^2 + 4)e^{3x}) \\ (d) \ f(t) = e^{t^3/\sin t}$$

Exercise 14

(a) Use the quotient rule to differentiate each of the functions below.

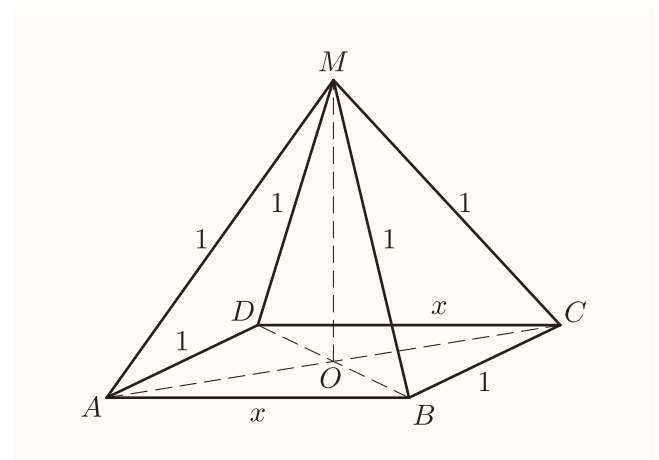
$$(i) \ f(v) = \frac{\tan v}{v} \quad (ii) \ g(x) = \frac{\sin x}{2x^3}$$

(b) By rewriting each of the functions in part (a) as a product, use the product rule to differentiate it and hence confirm your answer found in part (a).

3 More differentiation

Exercise 15

The pyramid-shaped structure below has all edges of length 1, except that the edges AB and CD can be any length x for which the structure can exist. The base of the structure is a rectangle.



(a) Show that the perpendicular height of the pyramid (length OM) is given by

$$OM = \frac{1}{2} \sqrt{3 - x^2}.$$

- (b) Hence show that the volume V of the pyramid is given by

$$V = \frac{1}{6}x\sqrt{3-x^2}.$$

To do this, you can use the fact that the volume V of a pyramid is given by the formula

$$V = \frac{1}{3}Ah,$$

where A is the area of the base and h is the perpendicular height.

- (c) Show that the maximum possible value of x is $\sqrt{3}$.
(d) Show that

$$\frac{dV}{dx} = \frac{3-2x^2}{6\sqrt{3-x^2}}.$$

- (e) Hence find the value of x that gives the maximum possible volume of the pyramid. (You can assume that the formula found in part (b) gives V as a function of x that is continuous on its whole domain and differentiable at every point in its domain that is not an endpoint.)
(f) Use your answer to part (e) to find the maximum possible volume of the pyramid.

Exercise 16

Use the module computer algebra system to plot the graph of the function that gives V in terms of x in Exercise 15 above. Does the graph support your answer to that exercise?

Exercise 17

- (a) Use the inverse function rule, the derivative of e^x and the constant multiple rule to show that the derivative of the function $y = \ln(2x)$ is $dy/dx = 1/x$.
(b) By rewriting the expression $\ln(2x)$ as the sum of two natural logarithms and differentiating the result, confirm your answer to part (b) above.

4 Reversing differentiation

Exercise 18

- (a) Use differentiation to show that the function $F(x) = 2x^{1/2}$ is an antiderivative of the function $f(x) = x^{-1/2}$.
(b) What is the indefinite integral of the function $f(x) = x^{-1/2}$?

Exercise 19

Find the indefinite integrals of the following functions.

- (a) $f(x) = x^{12}$ (b) $g(x) = x^{-4}$
(c) $h(x) = \frac{1}{x\sqrt{x}}$ (d) $k(x) = \frac{2}{3x^2}$

5 Particular antiderivatives

Exercise 20

A ball is thrown vertically upwards from a height of 1.6 metres above the ground and then allowed to fall. Assume that its velocity v (in ms^{-1}) is given by the equation

$$v = 15 - 10t,$$

where t is the time in seconds since it was thrown.

- (a) At what time does the ball stop rising?
(b) What is the distance above the ground at this time?

Exercise 21

An object moves with constant acceleration -10 m s^{-2} . Its initial velocity is 5 m s^{-1} . It stops moving when it returns to its starting point.

- Find a formula for its velocity v (in m s^{-1}) in terms of the time t (in seconds) since it began moving.
- Find a formula for its displacement s (in metres) from its starting point, in terms of t .
- At what time does the object return to its starting point?
- Use the computer algebra system to plot a graph of the formula that you found in part (b).
- Use your graph to estimate the greatest distance that the object reaches from its starting point.
- Use algebra to find the greatest distance that the object reaches from its starting point.

Exercise 23

Find the indefinite integrals of the following functions.

- $f(x) = \frac{3}{x}$
- $g(x) = \frac{2}{3x}$
- $h(x) = \frac{3x+1}{3x}$
- $k(x) = \frac{1}{2+2x^2}$
- $p(x) = \frac{2}{\sqrt{3-3x^2}}$

6 Antiderivatives of further standard functions

Exercise 22

Find the indefinite integrals of the following functions.

- $f(x) = e^{3x} - 2 \cos x$
- $g(x) = x^{3/2} + 4 \sin x$
- $h(x) = 2 \sec^2 x + 3x^2$

Solutions to exercises

Solution to Exercise 1

(a) $f(x) = \cos x - 2 \sin x$, so

$$f'(x) = -\sin x - 2 \cos x.$$

(b) $g(t) = t^2 + \tan t$, so

$$g'(t) = 2t + \sec^2 t.$$

(c) $h(a) = 3(1 + \cos a) = 3 + 3 \cos a$, so

$$h'(a) = -3 \sin a.$$

(d) $y = \frac{4}{x} + \sin x = 4x^{-1} + \sin x$, so

$$\begin{aligned} \frac{dy}{dx} &= -4x^{-2} + \cos x \\ &= -\frac{4}{x^2} + \cos x. \end{aligned}$$

Solution to Exercise 2

(a) $f(x) = \sin x + 2 \ln x$, so

$$\begin{aligned} f'(x) &= \cos x + 2 \times \frac{1}{x} \\ &= \cos x + \frac{2}{x}. \end{aligned}$$

(b) $g(t) = 2e^t + \cos t$, so

$$g'(t) = 2e^t - \sin t.$$

(c) $y = 5(2 \sin x - 3) = 10 \sin x - 15$, so

$$\frac{dy}{dx} = 10 \cos x.$$

(d) $w = \frac{x^2}{2} + 2 \tan x$, so

$$\frac{dw}{dx} = x + 2 \sec^2 x.$$

(e) $p(r) = 3 \left(\tan r + \frac{1}{r^2} \right)$
 $= 3 \tan r + 3r^{-2},$

so

$$\begin{aligned} p'(r) &= 3 \sec^2 r - 6r^{-3} \\ &= 3 \sec^2 r - \frac{6}{r^3}. \end{aligned}$$

Solution to Exercise 3

(a) $y = x^3 \ln x$, so, by the product rule,

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 \ln x + x^3 \times \frac{1}{x} \\ &= 3x^2 \ln x + x^2. \end{aligned}$$

(b) $m(x) = e^x \sin x$, so, by the product rule,

$$\begin{aligned} m'(x) &= e^x \sin x + e^x \cos x \\ &= e^x (\sin x + \cos x). \end{aligned}$$

(c) $v = t^2 \sin t$, so, by the product rule,

$$\begin{aligned} \frac{dv}{dt} &= 2t \sin t + t^2 \cos t \\ &= t(2 \sin t + t \cos t). \end{aligned}$$

Solution to Exercise 4

(a) $f(x) = (x^3 - 8x^{4/3} + 2) \cos x$, so, by the product rule,

$$\begin{aligned} f'(x) &= \left(3x^2 - 8 \times \frac{4}{3} x^{1/3} \right) \cos x \\ &\quad + (x^3 - 8x^{4/3} + 2)(-\sin x) \\ &= \left(3x^2 - \frac{32}{3} x^{1/3} \right) \cos x \\ &\quad - (x^3 - 8x^{4/3} + 2) \sin x. \end{aligned}$$

(b) $g(t) = (2t^4 + \sin t) \tan t$, so, by the product rule,

$$\begin{aligned} g'(t) &= (8t^3 + \cos t) \tan t \\ &\quad + (2t^4 + \sin t) \sec^2 t. \end{aligned}$$

(c) $h(x) = \left(5x^7 - x^2 + 5 - \frac{2}{x} \right) e^x$, so, by the product rule,

$$\begin{aligned} h'(x) &= (35x^6 - 2x + 2x^{-2})e^x \\ &\quad + \left(5x^7 - x^2 + 5 - \frac{2}{x} \right) e^x \\ &= \left(5x^7 + 35x^6 - x^2 - 2x + 5 - \frac{2}{x} + \frac{2}{x^2} \right) e^x \\ &= \frac{(5x^9 + 35x^8 - x^4 - 2x^3 + 5x^2 - 2x + 2)e^x}{x^2}. \end{aligned}$$

(d) $f(\theta) = \theta^6 \ln \theta$, so, by the product rule,

$$\begin{aligned} f'(\theta) &= 6\theta^5 \ln \theta + \theta^6 \times \frac{1}{\theta} \\ &= \theta^5 (6 \ln \theta + 1). \end{aligned}$$

(e) $y = (1 - x^3) \ln x$, so, by the product rule,

$$\begin{aligned} \frac{dy}{dx} &= (-3x^2) \ln x + (1 - x^3) \times \frac{1}{x} \\ &= -3x^2 \ln x + \frac{1}{x} - x^2 \\ &= \frac{1}{x} - x^2 (3 \ln x + 1) \\ &= \frac{1 - x^3 (3 \ln x + 1)}{x}. \end{aligned}$$

Solution to Exercise 5

$p(x) = \frac{e^x}{x^2} = e^x \times x^{-2}$, so, by the product rule,

$$\begin{aligned} p'(x) &= e^x \times x^{-2} + e^x \times -2x^{-3} \\ &= \frac{e^x}{x^2} - \frac{2e^x}{x^3} \\ &= \frac{xe^x - 2e^x}{x^3} \\ &= \frac{e^x(x-2)}{x^3}. \end{aligned}$$

Solution to Exercise 6

Here $y = x^3 \ln x$, so the product rule gives

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 \ln x + x^3 \times \frac{1}{x} \\ &= 3x^2 \ln x + x^2. \end{aligned}$$

So the gradient of the graph of $y = x^3 \ln x$ when $x = 1$ is

$$3 \times 1^2 \times \ln 1 + 1^2 = 0 + 1 = 1.$$

(Remember that $\ln 1 = 0$.)

Solution to Exercise 7

(a) $f(x) = \frac{x^7 + 1}{x^4 + 1}$, so, by the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(x^4 + 1) \times 7x^6 - (x^7 + 1) \times 4x^3}{(x^4 + 1)^2} \\ &= \frac{7x^{10} + 7x^6 - 4x^{10} - 4x^3}{(x^4 + 1)^2} \\ &= \frac{3x^{10} + 7x^6 - 4x^3}{(x^4 + 1)^2} \\ &= \frac{x^3(3x^7 + 7x^3 - 4)}{(x^4 + 1)^2}. \end{aligned}$$

(b) $g(x) = \frac{4x + e^x}{x^8 - 2}$, so, by the quotient rule,

$$\begin{aligned} g'(x) &= \frac{(x^8 - 2)(4 + e^x) - (4x + e^x)8x^7}{(x^8 - 2)^2} \\ &= \frac{4x^8 + x^8 e^x - 8 - 2e^x - 32x^8 - 8x^7 e^x}{(x^8 - 2)^2} \\ &= \frac{-28x^8 - 8 + e^x(x^8 - 8x^7 - 2)}{(x^8 - 2)^2}. \end{aligned}$$

(c) $y = \frac{\sin x}{e^x}$, so, by the quotient rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{e^x \cos x - (\sin x)e^x}{(e^x)^2} \\ &= \frac{\cos x - \sin x}{e^x}. \end{aligned}$$

(d) $m = \frac{\ln x}{x}$, so, by the quotient rule,

$$\begin{aligned} \frac{dm}{dx} &= \frac{x \times \frac{1}{x} - (\ln x) \times 1}{x^2} \\ &= \frac{1 - \ln x}{x^2}. \end{aligned}$$

Solution to Exercise 8

(a) $y = 3x - \frac{4}{x} + \frac{1}{x^2} = 3x - 4x^{-1} + x^{-2}$, so

$$\begin{aligned} \frac{dy}{dx} &= 3 + 4x^{-2} - 2x^{-3} \\ &= 3 + \frac{4}{x^2} - \frac{2}{x^3}. \end{aligned}$$

(b) Combining the fractions in the rule of the function gives

$$\begin{aligned} y &= \frac{3x^3}{x^2} - \frac{4x}{x^2} + \frac{1}{x^2} \\ &= \frac{3x^3 - 4x + 1}{x^2}. \end{aligned}$$

Using the quotient rule gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{x^2(9x^2 - 4) - (3x^3 - 4x + 1) \times 2x}{(x^2)^2} \\ &= \frac{9x^4 - 4x^2 - 6x^4 + 8x^2 - 2x}{x^4} \\ &= \frac{3x^4 + 4x^2 - 2x}{x^4} \\ &= \frac{3x^3 + 4x - 2}{x^3}. \end{aligned}$$

Expanding the fraction gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{3x^3}{x^3} + \frac{4x}{x^3} - \frac{2}{x^3} \\ &= 3 + \frac{4}{x^2} - \frac{2}{x^3}, \end{aligned}$$

which is the same answer as in part (a) above.

Solution to Exercise 9

(a) The derivative of $p(x) = (x^2 - 1)/(x^2 + 1)$ is

$$\begin{aligned} p'(x) &= \frac{(x^2 + 1)(2x) - (x^2 - 1)(2x)}{(x^2 + 1)^2} \\ &= \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2 + 1)^2} \\ &= \frac{4x}{(x^2 + 1)^2}. \end{aligned}$$

- (b) The derivative of

$$z(y) = \frac{2 \sin y}{y^2 - 4y + 3}$$

is

$$\begin{aligned} z'(y) &= \frac{(y^2 - 4y + 3)(2 \cos y) - (2 \sin y)(2y - 4)}{(y^2 - 4y + 3)^2} \\ &= \frac{2(y^2 - 4y + 3) \cos y - 4(y - 2) \sin y}{(y^2 - 4y + 3)^2}. \end{aligned}$$

Solution to Exercise 10

- (a) Here $y = e^u$, where $u = x^2/2$. So $dy/du = e^u$ and $du/dx = x$. Hence

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = xe^u = xe^{x^2/2}.$$

- (b) Here $y = \sin u$ where $u = x^3$. So $dy/du = \cos u$ and $du/dx = 3x^2$. Hence

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = \cos u \times 3x^2 \\ &= 3x^2 \cos(x^3). \end{aligned}$$

- (c) Here $y = \cos u$ where $u = \sqrt{x} = x^{1/2}$. So $dy/du = -\sin u$ and

$$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

Hence

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = -\sin u \times \frac{1}{2\sqrt{x}} \\ &= -\frac{1}{2\sqrt{x}} \sin \sqrt{x}. \end{aligned}$$

- (d) Here $y = \sqrt{u} = u^{1/2}$ where $u = \cos x$. So $dy/du = \frac{1}{2}u^{-1/2}$ and $du/dx = -\sin x$. Hence

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= \frac{1}{2}u^{-1/2}(-\sin x) \\ &= -\frac{1}{2}(\cos x)^{-1/2} \sin x \\ &= -\frac{\sin x}{2\sqrt{\cos x}}. \end{aligned}$$

- (e) Here $y = \tan u$ where $u = \ln x$. So $dy/du = \sec^2 u$ and $du/dx = 1/x$. Hence

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = \sec^2 u \times \frac{1}{x} \\ &= \frac{\sec^2(\ln x)}{x}. \end{aligned}$$

Solution to Exercise 11

- (a) Here $y = \sqrt{3x^2 + 2x + 1} = \sqrt{u} = u^{1/2}$, where $u = 3x^2 + 2x + 1$. So

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} \quad \text{and} \quad \frac{du}{dx} = 6x + 2.$$

Hence

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = \left(\frac{1}{2\sqrt{u}} \right) (6x + 2) \\ &= \frac{6x + 2}{2\sqrt{3x^2 + 2x + 1}} \\ &= \frac{3x + 1}{\sqrt{3x^2 + 2x + 1}}. \end{aligned}$$

- (b) Here $w = e^{-r^2} = e^u$, where $u = -r^2$. So

$$\frac{dw}{du} = e^u \quad \text{and} \quad \frac{du}{dr} = -2r.$$

Hence

$$\frac{dw}{dr} = \frac{dw}{du} \frac{du}{dr} = e^u \times (-2r) = -2re^{-r^2}.$$

Solution to Exercise 12

- (a) Using the product rule, where

$f(x) = e^{3x} \sin(2x)$, we have

$$\begin{aligned} f'(x) &= (3e^{3x})(\sin(2x)) + (e^{3x})(2 \cos(2x)) \\ &= e^{3x}(3 \sin(2x) + 2 \cos(2x)). \end{aligned}$$

- (b) Using the quotient rule, where

$g(t) = 3t^2 / \ln(4t)$, we have

$$\begin{aligned} g'(t) &= \frac{(\ln(4t))(6t) - (3t^2)(1/t)}{(\ln(4t))^2} \\ &= \frac{6t \ln(4t) - 3t}{(\ln(4t))^2}. \end{aligned}$$

- (c) Using the product rule, where $y = (x^3 + 2x - 1) \cos(5x)$, we have

$$\begin{aligned} \frac{dy}{dx} &= (3x^2 + 2) \cos(5x) \\ &\quad + (x^3 + 2x - 1)(-5 \sin(5x)) \\ &= (3x^2 + 2) \cos(5x) \\ &\quad - 5(x^3 + 2x - 1) \sin(5x). \end{aligned}$$

- (d) Here $z = \sin(e^{3t}) = \sin u$, where $u = e^{3t}$. By the chain rule,

$$\begin{aligned} \frac{dz}{dt} &= \frac{dz}{du} \frac{du}{dt} \\ &= (\cos u)(3e^{3t}) \\ &= \cos(e^{3t})(3e^{3t}) = 3e^{3t} \cos(e^{3t}). \end{aligned}$$

Solution to Exercise 13

(a) Here $f(t) = \cos^2(3t) \ln(5t)$, so

$$\begin{aligned} f'(t) &= \frac{d}{dt} (\cos^2(3t)) \ln(5t) \\ &\quad + \cos^2(3t) \frac{d}{dt} (\ln(5t)) \\ &\quad \text{(by the product rule)} \\ &= 2 \cos(3t)(-3 \sin(3t)) \ln(5t) \\ &\quad + \cos^2(3t) \frac{1}{t} \\ &\quad \text{(by the chain rule)} \\ &= \cos(3t) \left(\frac{\cos(3t)}{t} - 6 \sin(3t) \ln(5t) \right). \end{aligned}$$

(b) Here $f(x) = \frac{e^{2x}}{(x^2 + 1)^2}$, so

$$\begin{aligned} f'(x) &= \frac{(x^2 + 1)^2 \frac{d}{dx} (e^{2x}) - e^{2x} \frac{d}{dx} ((x^2 + 1)^2)}{(x^2 + 1)^4} \\ &\quad \text{(by the quotient rule)} \\ &= \frac{(x^2 + 1)^2 \times 2e^{2x} - e^{2x} \times 2(x^2 + 1)2x}{(x^2 + 1)^4} \\ &\quad \text{(by the chain rule)} \\ &= \frac{2e^{2x}(x^2 + 1)(x^2 + 1 - 2x)}{(x^2 + 1)^4} \\ &= \frac{2e^{2x}(x - 1)^2}{(x^2 + 1)^3}. \end{aligned}$$

(c) Here $f(x) = \sin((x^2 + 4)e^{3x})$, so

$$\begin{aligned} f'(x) &= \cos((x^2 + 4)e^{3x}) \frac{d}{dx} ((x^2 + 4)e^{3x}) \\ &\quad \text{(by the chain rule)} \\ &= \cos((x^2 + 4)e^{3x}) (2xe^{3x} + (x^2 + 4)3e^{3x}) \\ &\quad \text{(by the product rule)} \\ &= e^{3x} (3x^2 + 2x + 12) \cos((x^2 + 4)e^{3x}). \end{aligned}$$

(d) Here $f(t) = e^{t^3/\sin t}$, so

$$\begin{aligned} f'(t) &= e^{t^3/\sin t} \frac{d}{dt} \left(\frac{t^3}{\sin t} \right) \\ &\quad \text{(by the chain rule)} \\ &= e^{t^3/\sin t} \left(\frac{\sin t \times 3t^2 - t^3 \times \cos t}{\sin^2 t} \right) \\ &\quad \text{(by the quotient rule)} \\ &= \left(\frac{3 \sin t - t \cos t}{\sin^2 t} \right) t^2 e^{t^3/\sin t}. \end{aligned}$$

Solution to Exercise 14

(a) (i) $f(v) = \frac{\tan v}{v}$, so

$$f'(v) = \frac{v \sec^2 v - \tan v}{v^2}.$$

(ii) $g(x) = \frac{\sin x}{2x^3}$, so

$$\begin{aligned} g'(x) &= \frac{2x^3 \cos x - \sin x \times 6x^2}{(2x^3)^2} \\ &= \frac{2x^2(x \cos x - 3 \sin x)}{4x^6} \\ &= \frac{x \cos x - 3 \sin x}{2x^4}. \end{aligned}$$

(b) (i) $f(v) = \frac{\tan v}{v} = v^{-1} \tan v$. So

$$\begin{aligned} f'(v) &= v^{-1} \sec^2 v + (-v^{-2}) \tan v \\ &= v^{-1} \sec^2 v - v^{-2} \tan v \\ &= \frac{v \sec^2 v - \tan v}{v^2}. \end{aligned}$$

(ii) $g(x) = \frac{\sin x}{2x^3} = (2x^3)^{-1} \times \sin x$. So

$$\begin{aligned} g'(x) &= -(2x^3)^{-2} (6x^2) \sin x \\ &\quad + (2x^3)^{-1} \times \cos x \\ &= \frac{-6x^2 \sin x + 2x^3 \cos x}{4x^6} \\ &= \frac{-3 \sin x + x \cos x}{2x^4} \\ &= \frac{x \cos x - 3 \sin x}{2x^4}. \end{aligned}$$

Solution to Exercise 15

(a) By Pythagoras' theorem for the right-angled triangle ABC , we have

$$AC = \sqrt{x^2 + 1^2} = \sqrt{x^2 + 1},$$

so

$$AO = \frac{1}{2} \sqrt{x^2 + 1}.$$

Hence, by Pythagoras' theorem for the right-angled triangle AOM , we have

$$\left(\frac{1}{2} \sqrt{x^2 + 1} \right)^2 + OM^2 = 1^2$$

$$\frac{1}{4}(x^2 + 1) + OM^2 = 1$$

$$OM^2 = 1 - \frac{1}{4}(x^2 + 1)$$

$$OM^2 = \frac{1}{4}(4 - x^2 - 1)$$

$$OM^2 = \frac{1}{4}(3 - x^2)$$

$$OM = \frac{1}{2} \sqrt{3 - x^2},$$

as required.

Exercise Booklet 7

- (b) The area of the base of the pyramid is

$$1 \times x = x.$$

Hence

$$V = \frac{1}{3}x \times \frac{1}{2}\sqrt{3-x^2};$$

that is,

$$V = \frac{1}{6}x\sqrt{3-x^2},$$

as required.

- (c) The maximum possible value of x occurs when the height of the pyramid is zero.

This gives

$$\frac{1}{2}\sqrt{3-x^2} = 0$$

$$3-x^2 = 0$$

$$x^2 = 3.$$

Hence, since x is a length and therefore positive,

$$x = \sqrt{3}.$$

- (d) We have

$$V = \frac{1}{6}x\sqrt{3-x^2}.$$

So, by the product rule,

$$\begin{aligned}\frac{dV}{dx} &= \frac{1}{6} \left(x \frac{d}{dx} \sqrt{3-x^2} + (\sqrt{3-x^2}) \times 1 \right) \\ &= \frac{1}{6} \left(x \frac{d}{dx} (3-x^2)^{1/2} + \sqrt{3-x^2} \right).\end{aligned}$$

Hence, by the chain rule,

$$\begin{aligned}\frac{dV}{dx} &= \frac{1}{6} \left(x \times \frac{1}{2}(3-x^2)^{-1/2} \times (-2x) \right. \\ &\quad \left. + \sqrt{3-x^2} \right) \\ &= \frac{1}{6} \left(\frac{-x^2}{\sqrt{3-x^2}} + \sqrt{3-x^2} \right) \\ &= \frac{1}{6} \left(\frac{-x^2 + 3 - x^2}{\sqrt{3-x^2}} \right) \\ &= \frac{3-2x^2}{6\sqrt{3-x^2}}.\end{aligned}$$

- (e) We have to find the value of x , between 0 and $\sqrt{3}$, that gives the maximum value of V .

The function

$$V = \frac{1}{6}x\sqrt{3-x^2} \quad (x \in [0, \sqrt{3}])$$

is continuous on its domain $[0, \sqrt{3}]$ and differentiable at every value of x in this interval except possibly the endpoints.

So the maximum value of V must occur either at an endpoint of the interval $[0, \sqrt{3}]$, or at a stationary point.

We have

$$\frac{dV}{dx} = \frac{3-2x^2}{6\sqrt{3-x^2}}.$$

At a stationary point, $dV/dx = 0$, which gives

$$\frac{3-2x^2}{6\sqrt{3-x^2}} = 0$$

$$3-2x^2 = 0$$

$$x^2 = \frac{3}{2}.$$

Hence, since $x \geq 0$,

$$x = \sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{1}{2}\sqrt{6}.$$

So the only stationary point is $\frac{1}{2}\sqrt{6}$, and this value does lie in the interval $[0, \sqrt{3}]$.

So the maximum value of V occurs when $x = 0$, $x = \frac{1}{2}\sqrt{6}$ or $x = \sqrt{3}$.

When $x = 0$ the width of the pyramid is zero, and when $x = \sqrt{3}$ its height is zero, so in both these cases the volume V will be zero. When $x = \frac{1}{2}\sqrt{6}$, the length, width and height of the pyramid are all positive, so the volume V will also be positive.

So the maximum volume is achieved when $x = \frac{1}{2}\sqrt{6}$.

- (f) Substituting $x = \frac{1}{2}\sqrt{6}$ into the formula

$$V = \frac{1}{6}x\sqrt{3-x^2}$$

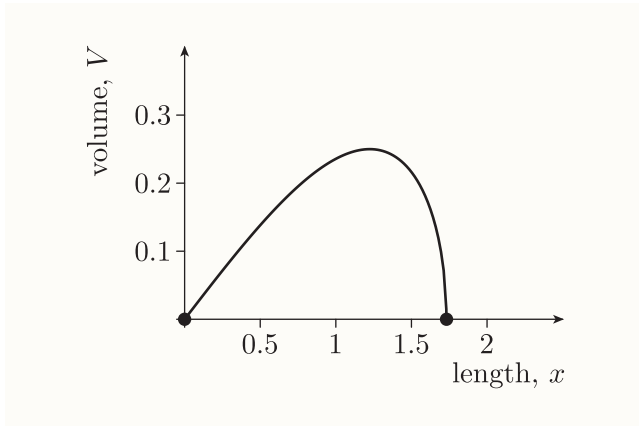
gives

$$\begin{aligned}V &= \frac{1}{6} \times \frac{1}{2}\sqrt{6} \sqrt{3 - (\frac{1}{2}\sqrt{6})^2} \\ &= \frac{1}{12}\sqrt{6} \sqrt{3 - \frac{1}{4} \times 6} \\ &= \frac{1}{12}\sqrt{6} \sqrt{\frac{3}{2}} \\ &= \frac{1}{12}\sqrt{2} \sqrt{3} \frac{\sqrt{3}}{\sqrt{2}} \\ &= \frac{1}{12} \times 3 \\ &= \frac{1}{4}.\end{aligned}$$

So the maximum volume is $\frac{1}{4}$.

Solution to Exercise 16

The graph of the function from Exercise 15 is shown below. The stationary point seems to occur at about $x = 1.25$, which supports the answer found in the solution to part (e) of Exercise 15, since $\frac{1}{2}\sqrt{6} = 1.224\dots$. Also, the maximum value of V seems to be about 0.25, which supports the answer found in the solution to part (f) of Exercise 15.



Solution to Exercise 17

- (a) Here $y = \ln 2x$. This gives $2x = e^y$, so $x = \frac{1}{2}e^y$. Hence

$$\frac{dx}{dy} = \frac{1}{2}e^y.$$

Therefore, by the inverse function rule,

$$\frac{dy}{dx} = \frac{1}{\frac{1}{2}e^y} = \frac{1}{x}.$$

- (b) We have $\ln(2x) = \ln x + \ln 2$. Hence, since $\ln 2$ is a constant, the derivative of $\ln(2x)$ is the same as the derivative of $\ln x$, which is $1/x$. So

$$\frac{dy}{dx} = \frac{1}{x},$$

which confirms the answer to part (b).

Solution to Exercise 18

- (a) Since

$$\frac{d}{dx}(2x^{1/2}) = x^{-1/2},$$

it follows that the function $F(x) = 2x^{1/2}$ is an antiderivative of the function $f(x) = x^{-1/2}$.

- (b) The indefinite integral of the function $f(x) = x^{-1/2}$ is

$$F(x) = 2x^{1/2} + c,$$

where c is an arbitrary constant.

Solution to Exercise 19

- (a) The indefinite integral of $f(x) = x^{12}$ is

$$F(x) = \frac{1}{13}x^{13} + c.$$

- (b) The indefinite integral of $g(x) = x^{-4}$ is

$$\begin{aligned} G(x) &= \frac{1}{-3}x^{-3} + c \\ &= -\frac{1}{3}x^{-3} + c \\ &= -\frac{1}{3x^3} + c. \end{aligned}$$

(Either of the final two expressions above is acceptable as the final answer. Although the final expression is written without a negative index, which is often preferable, the rule of the function in the question itself contains a negative index, so you might prefer to leave the final answer also containing a negative index.)

- (c) The indefinite integral of

$$h(x) = \frac{1}{x\sqrt{x}} = \frac{1}{x^{3/2}} = x^{-3/2}$$

is

$$\begin{aligned} H(x) &= \frac{1}{-1/2}x^{-1/2} + c \\ &= -\frac{2}{x^{1/2}} + c \\ &= -\frac{2}{\sqrt{x}} + c. \end{aligned}$$

(Either of the final two expressions above is acceptable as the final answer, but the final expression is probably preferable since the rule of the function in the question contains a square root sign rather than a fractional index.)

- (d) The indefinite integral of

$$k(x) = \frac{2}{3x^2} = \frac{2}{3}x^{-2}$$

is

$$\begin{aligned} K(x) &= \frac{2}{3} \times \frac{1}{-1}x^{-1} + c \\ &= -\frac{2}{3}x^{-1} + c \\ &= -\frac{2}{3x} + c. \end{aligned}$$

(Either of the final two expressions above is acceptable as the final answer, but the final expression is probably preferable since it doesn't contain a negative index.)

Solution to Exercise 20

- (a) At the moment when the ball stops rising, $v = 0$. Since $v = 15 - 10t$, this gives

$$15 - 10t = 0$$

$$10t = 15$$

$$t = 1.5.$$

So the ball stops rising at time $t = 1.5$ seconds.

- (b) Let s be the displacement (in metres) of the ball above the ground.

Integrating the equation for v gives

$$s = 15t - 10 \times \frac{1}{2}t^2 + c;$$

that is,

$$s = 15t - 5t^2 + c,$$

where c is a constant.

But when $t = 0$, $s = 1.6$. This gives

$$1.6 = 15 \times 0 - 5 \times 0^2 + c;$$

that is, $c = 1.6$. Hence the equation for s in terms of t is

$$s = 15t - 5t^2 + 1.6.$$

The distance above the ground at the time when the ball stops rising, that is, at time $t = 1.5$, is therefore given by

$$\begin{aligned} s &= 15 \times 1.5 - 5 \times (1.5)^2 + 1.6 \\ &= 12.85. \end{aligned}$$

So the required distance is 12.85 metres.

(This question is based on the assumption that the ball's initial velocity is 15 m s^{-1} upwards, and the acceleration due to gravity is 10 m s^{-2} downwards. In Units 6–8 we usually take the acceleration due to gravity to be 9.8 m s^{-2} downwards.)

Solution to Exercise 21

- (a) The equation for a in terms of t is

$$a = -10.$$

Integrating gives

$$v = -10t + c,$$

where c is a constant.

But $v = 5$ when $t = 0$, which gives

$$5 = -10 \times 0 + c,$$

so $c = 5$. Hence

$$v = -10t + 5.$$

- (b) Integrating the equation for v gives

$$s = -10 \times \frac{1}{2}t^2 + 5t + k;$$

that is,

$$s = -5t^2 + 5t + k,$$

where k is a constant.

Since $s = 0$ when $t = 0$, it follows that $k = 0$.

So

$$s = -5t^2 + 5t.$$

- (c) When the object returns to its starting point, $s = 0$. Substituting into the equation from part (b) gives

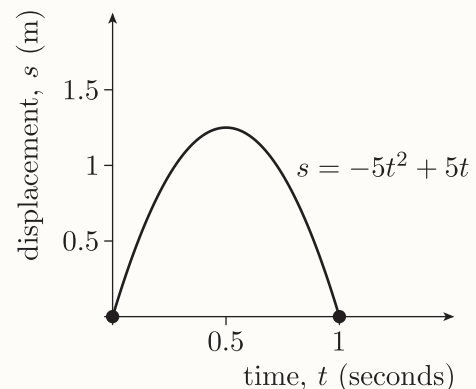
$$0 = -5t^2 + 5t$$

$$5t(1 - t) = 0$$

$$t = 0 \quad \text{or} \quad t = 1.$$

So the object returns to its starting point one second after it started moving.

- (d) A graph of the formula from part (b) is shown below.



- (e) The greatest distance that the object reaches from its starting point appears to be about 1.25 m.

- (f) The greatest distance occurs when $v = 0$. Substituting this value of v into the equation found in part (a) gives

$$0 = -10t + 5$$

$$10t = 5$$

$$t = 0.5.$$

Substituting this value of t into the equation found in part (b) gives

$$\begin{aligned} s &= -5 \times 0.5^2 + 5 \times 0.5 \\ &= -1.25 + 2.5 = 1.25. \end{aligned}$$

So the greatest distance is 1.25 m.

(An alternative way to find that the greatest distance is reached when $t = 0.5$ is to use the fact that the graph in part (d) is a parabola with x -intercepts 0 and 1, so its vertex has x -coordinate $\frac{1}{2}(0 + 1) = 0.5$.

- (d) The indefinite integral of

$$k(x) = \frac{1}{2 + 2x^2} = \frac{1}{2} \times \frac{1}{1 + x^2}$$

$$\text{is } K(x) = \frac{1}{2} \tan^{-1} x + c.$$

- (e) The function is

$$p(x) = \frac{2}{\sqrt{3 - 3x^2}},$$

which can be written as

$$p(x) = \frac{2}{\sqrt{3}\sqrt{1 - x^2}} = \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{1 - x^2}}.$$

Hence its indefinite integral is

$$P(x) = \frac{2}{\sqrt{3}} \times \sin^{-1} x + c.$$

Solution to Exercise 22

- (a) The indefinite integral of

$$f(x) = e^{3x} - 2 \cos x \text{ is}$$

$$F(x) = \frac{1}{3}e^{3x} - 2 \sin x + c.$$

- (b) The indefinite integral of

$$g(x) = x^{3/2} + 4 \sin x \text{ is}$$

$$G(x) = \frac{2}{5}x^{5/2} - 4 \cos x + c.$$

- (c) The indefinite integral of

$$h(x) = 2 \sec^2 x + 3x^2 \text{ is}$$

$$H(x) = 2 \tan x + x^3 + c.$$

Solution to Exercise 23

- (a) The indefinite integral of

$$f(x) = \frac{3}{x} = 3 \times \frac{1}{x}$$

$$\text{is } F(x) = 3 \ln |x| + c.$$

- (b) The indefinite integral of

$$g(x) = \frac{2}{3x} = \frac{2}{3} \times \frac{1}{x}$$

$$\text{is } G(x) = \frac{2}{3} \ln |x| + c.$$

- (c) The indefinite integral of

$$h(x) = \frac{3x + 1}{3x} = 1 + \frac{1}{3x} = 1 + \frac{1}{3} \times \frac{1}{x}$$

$$\text{is } H(x) = x + \frac{1}{3} \ln |x| + c.$$